VECTOR-TENSOR MULTIPLET IN $N = 2$ SUPERSPACE WITH CENTRAL CHARGE

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Abstract. We use the four-dimensional $N = 2$ central charge superspace to give a geometrical construction of the Abelian vector-tensor multiplet consisting, under $N = 1$ supersymmetry, of one vector and one linear multiplet. We derive the component field supersymmetry and central charge transformations, and show that there is a super-Lagrangian, the higher components of which are all total derivatives, allowing us to construct superfield and component actions.

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1. Introduction

It has long been known that there is an $N = 2$ supermultiplet which, with respect to $N = 1$ supersymmetry, appears as a combination of one vector and one chiral multiplet [1-4]. This supermultiplet is irreducible under $N = 2$ supersymmetry and carries vanishing central charge. It is usually referred to as the vector supermultiplet. Not long after its introduction, it was realized that there exists a variant, the vector-tensor multiplet, which, under $N = 1$ supersymmetry, appears as a combination of one vector and one linear multiplet [5, 6]. This variant is irreducible under $N = 2$ and, in contrast, carries non-vanishing central charge off-shell. On shell it is equivalent to the vector multiplet. This multiplet was originally constructed using component field techniques [5, 6], and it and its properties have received little attention. Recently, however, within the context of trying to understand the consequences of $N = 2$ duality in superstrings, the vector-tensor multiplet has re-emerged. Specifically, when heterotic string theory is reduced to an $N = 2$ theory in four dimensions the dilaton and the antisymmetric tensor field lie in a vector-tensor multiplet [7]. It is clear from this work that the vector-tensor supermultiplet is fundamental in heterotic theories and that elucidation of its properties, such as its behavior under duality transformations, is of importance. The coupling of the vector-tensor multiplet to supergravity has also recently been considered [8]. It was shown that this requires the gauging of the central charge, leading to a Chern-Simons coupling between the vector-tensor multiplet and a vector multiplet.

If one wants to fully understand the complete, off-shell structure of the tensor-vector multiplet, one powerful approach is to construct the appropriate $N = 2$ superfield. It is the purpose of this paper to provide such a formulation. Much of the interesting structure of

the vector-tensor multiplet appears in its couplings to other multiplets [8]. Here, however, we will restrict ourselves to a superfield formulation of the free Abelian case, leaving interacting generalization to future publications. Since the vector-tensor supermultiplet has non-zero central charge, it is necessary to expand the usual superspace. The maximal central extension of the $N = 2$ superalgebra has two central charges. Consequently the corresponding superspace has two extra bosonic coordinates [9]. Working in this superspace, one can introduce superfields and covariant constraints. Following the geometrical formulation of supergauge fields, we introduce a super-connection and the associated curvature. We then introduce the appropriate constraints, solve the Bianchi identities and show how the component fields of the tensor-vector multiplet emerge. Using superspace techniques we rederive the supersymmetry and central charge transformations and, after showing that there exist a Lagrangian superfield in central charge superspace, the higher components of which are all total derivatives, we give the superfield and component actions. The existence of a superfield Lagrangian is an example of a central-charge generalization of the superactions described in [10]. It suggests that there is an even-dimensional submanifold of the central charge superspace naturally associated with the vector-tensor multiplet.

Central charge superspace has recently also been considered in [11, 12] and used to rederive the usual gauge supermultiplet. Also, we want to point out that there is a strong relationship between the theory of $N = 1$ supersymmetry in six dimensions and four-dimensional, $N = 2$ central charge superspace. Indeed, the superfield equations of motion in the six-dimensional theory motivated the choice of one important constraint equation in four-dimensional central charge superspace. In this sense, our constraint can be understood as a superfield realization of the dimensional reduction by Legendre transformation discussed in [6].

2. $N = 2$ Superspace

We start by briefly recapitulating the relevant formulae of $N = 2$ superspace. Throughout the paper we will use the conventions of [9, 4, 13]. The $N = 2$ supersymmetry algebra is obtained from the Poincaré algebra by adding four fermionic operators Q^i_α ($\alpha = 1, 2$ and $i = 1, 2$) and their anti-hermitian conjugates $\bar{Q}_{\dot{\alpha} i} = -(Q^i_\alpha)^\dagger$. Moreover the algebra admits one complex "central charge" $Z = Z_1 + iZ_2$ where Z_1 and Z_2 are hermitian. These operators satisfy the following anticommutation relations:

$$\begin{aligned} \{Q^i_\alpha, \bar{Q}_{\dot{\alpha} j}\} &= 2\delta^i{}_j \sigma^m{}_{\alpha\dot{\alpha}} P_m, \\ \{Q^i_\alpha, Q^j_\beta\} &= 2\epsilon_{\alpha\beta}\epsilon^{ij} Z, \\ \{\bar{Q}_{\dot{\alpha} i}, \bar{Q}_{\dot{\beta} j}\} &= -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ij} \bar{Z}, \end{aligned} \tag{2.1}$$

where P_m are the four-momenta and $\bar{Z} = Z^\dagger$. The automorphism group of algebra (2.1) is $\mathrm{SL}(2, \mathbb{C}) \otimes \mathrm{SU}(2)$ where the SU(2) acts on the i index in Q^i_α. The antisymmetric tensor ϵ_{ij}

with $\epsilon_{12} = -1$ provides an invariant metric for raising and lowering the SU(2) indices by $a_i = \epsilon_{ij} a^j$ and $a^i = \epsilon^{ij} a_j$.

$N = 2$ superspace is a space with coordinates $z^M = (x^m, \theta_i^\alpha, \bar{\theta}^i_{\dot{\alpha}}, z, \bar{z})$ where x^m, z, and $\bar{z} = z^*$ are commuting bosonic coordinates while θ_i^α and $\bar{\theta}^i_{\dot{\alpha}} = (\theta_{\alpha i})^*$ are anticommuting fermionic coordinates. A superfield ϕ is a function of the superspace coordinates, $\phi = \phi(x^m, \theta_i^\alpha, \bar{\theta}^i_{\dot{\alpha}}, z, \bar{z})$. Taylor-series expansion of a general superfield ϕ in the θ coordinates terminates after a finite number of terms due to the anticommuting nature of θ. On the other hand, the expansion in z and $\bar{z}$ never ends. This means there are an infinite number of component fields (functions of the spacetime coordinates x^m) in a general superfield. This infinite number of component fields can be reduced to a finite number either off-shell, by applying appropriate constraints, or on-shell, by choosing equations of motion which propagate only a finite number of component fields.

Translations in the superspace are generated by the supercovariant differential operators ∂_a, ∂_z, $\partial_{\bar{z}}$, and

$$\begin{aligned} D^i_\alpha &= \frac{\partial}{\partial \theta_i^\alpha} + i\sigma^a{}_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha} i} \partial_a - i\theta^i_\alpha \partial_z, \\ \bar{D}_{\dot{\alpha} i} &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha} i}} - i\theta_i^\alpha \sigma^a{}_{\alpha\dot{\alpha}} \partial_a - i\bar{\theta}_{\dot{\alpha} i} \partial_{\bar{z}}, \end{aligned} \tag{2.2}$$

where $\bar{D}_{\dot{\alpha} i} = -(D^i_\alpha)^\dagger$. It is straightforward to compute the anticommutation relations for these operators. They are

$$\begin{aligned} \{D^i_\alpha, \bar{D}_{\dot{\alpha} j}\} &= -2i\delta^i{}_j \sigma^m{}_{\alpha\dot{\alpha}} \partial_m, \\ \{D^i_\alpha, D^j_\beta\} &= -2i\epsilon_{\alpha\beta}\epsilon^{ij}\partial_z, \\ \{\bar{D}_{\dot{\alpha} i}, \bar{D}_{\dot{\beta} j}\} &= 2i\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ij}\partial_{\bar{z}}. \end{aligned} \tag{2.3}$$

By construction D^i_α and $\bar{D}_{\dot{\alpha} i}$ anticommute with Q^i_α and $\bar{Q}_{\dot{\alpha} i}$ and so can be used to impose supersymmetric covariant conditions on superfields.

The supervielbein $e_A{}^M$ of the $N = 2$ superspace is defined as the matrix that relates the supercovariant derivatives $D_A = (\partial_a, D^\alpha_i, \bar{D}_{\dot{\alpha} i}, \partial_z, \partial_{\bar{z}})$ and the ordinary partial derivatives

$$D_A = e_A{}^M \frac{\partial}{\partial z^M}. \tag{2.4}$$

The matrix $e_M{}^A$ is the inverse of $e_A{}^M$. These two matrices define the geometry of the $N = 2$ superspace. The torsion T^A is defined as the exterior derivative of the supervielbein one-form $e^A = dz^M e_M{}^A$,

$$T^A = de^A = \tfrac{1}{2} e^C e^B T_{BC}{}^A. \tag{2.5}$$

The non-vanishing components of the torsion are found to be

$$\begin{aligned}
T_{\alpha\dot\alpha}^{i\,j\,a} &= T_{\dot\alpha\alpha}^{j\,i\,a} = -2i\epsilon^{ij}\sigma^a{}_{\alpha\dot\alpha},\\
T_{\alpha\beta}^{i\,j\,z} &= T_{\beta\alpha}^{j\,i\,z} = 2i\epsilon^{ij}\epsilon_{\alpha\beta},\\
T_{\dot\alpha\dot\beta}^{i\,j\,\bar z} &= T_{\dot\beta\dot\alpha}^{j\,i\,\bar z} = 2i\epsilon^{ij}\epsilon_{\dot\alpha\dot\beta}.
\end{aligned} \tag{2.6}$$

3. Vector-Tensor Multiplet in $N=2$ Superspace

In this section we give a geometrical superfield formulation of the vector-tensor multiplet. We begin by considering the usual geometrical form of super-gauge theory, though in superspace with central charge. We then find a suitable set of constraints on the superfield strength to reproduce the field content of the vector-tensor multiplet. Restricting our attention to Abelian gauge theories, let us introduce a hermitian connection $A = dz^M A_M = e^A A_A$. The hermiticity of the connection implies

$$A = dx^a A_a + d\theta_i^\alpha A_\alpha^i + d\bar\theta_{\dot\alpha}^i \bar A_i^{\dot\alpha} + dz A_z + d\bar z A_{\bar z}, \tag{3.1}$$

where A_a is real, $\bar A_i^{\dot\alpha} = (A^{\alpha i})^\dagger$, and $A_{\bar z} = A_z^\dagger$. The curvature two-form is defined as

$$F = dA = \tfrac{1}{2} e^B e^A F_{AB}. \tag{3.2}$$

The coefficient functions F_{BA} comprise thirteen Lorentz-covariant types. The ones that contain torsion terms are given by

$$\begin{aligned}
F_{\alpha\dot\alpha}^{i\,j} &= D_\alpha^i A_{\dot\alpha}^j + \bar D_{\dot\alpha}^j \bar A_\alpha^i - 2i\epsilon^{ij}\sigma^a{}_{\alpha\dot\alpha} A_a,\\
F_{\alpha\beta}^{i\,j} &= D_\alpha^i A_\beta^j + D_\beta^i A_\alpha^j + 2i\epsilon_{\alpha\beta}\epsilon^{ij} A_z,\\
F_{\dot\alpha\dot\beta}^{i\,j} &= \bar D_{\dot\alpha}^i \bar A_{\dot\beta}^j + \bar D_{\dot\beta}^i \bar A_{\dot\alpha}^j + 2i\epsilon_{\dot\alpha\dot\beta}\epsilon^{ij} A_{\bar z},
\end{aligned} \tag{3.3}$$

The curvature tensor F is subject to the Bianchi identities $dF=0$ or equivalently

$$D_A F_{BC} + T_{AB}{}^D F_{DC} + \text{graded cycle} = 0. \tag{3.4}$$

Each tensor component of the curvature tensor is an $N=2$ superfield which, in turn, has an infinite number of component fields. All but a finite number of these component fields must be eliminated by virtue of appropriate constraints. These constraints must be Lorentz, gauge, and supersymmetric covariant. It is natural to adopt a set of constraints which set the pure spinorial part of the curvature tensor to zero. That is

$$F_{\alpha\beta}^{i\,j} = F_{\alpha\dot\beta}^{i\,j} = F_{\dot\alpha\dot\beta}^{i\,j} = 0. \tag{3.5}$$

Before imposing further conditions we would like to explore the consequences of (3.5). To do so we must solve the Bianchi identities subject to these constraints. The result is that all the components of the curvature tensor F_{AB} are determined in terms of a single superfield

$F^i_{\alpha\bar{z}}$ and its hermitian conjugate $F_{\dot{\alpha}iz} = (F^i_{\alpha\bar{z}})^\dagger$. Henceforth, we denote these superfields by W^i_α and $\bar{W}_{\dot{\alpha}i}$. In particular $F^i_{\alpha z} = 0$ and

$$F_{ab} = -\tfrac{1}{16} i\epsilon^{ij} \bar{\sigma}_a{}^{\dot{\alpha}\alpha} \bar{\sigma}_b{}^{\dot{\beta}\beta} \left(\epsilon_{\dot{\alpha}\dot{\beta}} D^j_\beta W^i_\alpha + \epsilon_{\alpha\beta} \bar{D}^j_{\dot{\beta}} \bar{W}^i_{\dot{\alpha}} \right). \tag{3.6}$$

$$F^i_{\dot{\alpha}a} = -\tfrac{1}{2} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\sigma}_a{}^{\dot{\beta}\alpha} W^i_\alpha, \tag{3.7}$$

$$F_{a\bar{z}} = -\tfrac{1}{8} i\epsilon_{ij} \bar{\sigma}_a{}^{\dot{\beta}\alpha} \bar{D}^j_{\dot{\beta}} W^i_\alpha, \tag{3.8}$$

$$F_{z\bar{z}} = \tfrac{1}{4} i\epsilon_{ij} \epsilon^{\alpha\beta} D^i_\alpha W^j_\beta, \tag{3.9}$$

Furthermore, W^i_α is constrained to satisfy

$$D^{(j}_{(\beta} W^{i)}_{\alpha)} = 0, \qquad \bar{D}^{(j}_{(\dot{\beta}} \bar{W}^{i)}_{\dot{\alpha})} = 0, \tag{3.10}$$

$$\bar{D}^{(j}_{\dot{\beta}} W^{i)}_\alpha = 0, \qquad D^{(j}_\beta \bar{W}^{i)}_{\dot{\alpha}} = 0, \tag{3.11}$$

$$\epsilon^{\alpha\beta} D^{[i}_\alpha W^{j]}_\beta = -\epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}^{[i}_{\dot{\alpha}} \bar{W}^{j]}_{\dot{\beta}}, \tag{3.12}$$

$$\epsilon^{\alpha\beta} D^{(i}_\alpha W^{j)}_\beta = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}^{(i}_{\dot{\alpha}} \bar{W}^{j)}_{\dot{\beta}}. \tag{3.13}$$

Let us explore the consequences of (3.10)–(3.13) on the field content of W^i_α. The expansion of W^i_α in the anti-commuting coordinates has the general form

$$W^i_\alpha = \lambda^i_\alpha + \theta^\beta_j G^{i\,j}_{\alpha\beta} + \bar{\theta}^{\dot{\alpha}}_j H^{i\,j}_{\alpha\dot{\alpha}} + \mathcal{O}(\theta^2). \tag{3.14}$$

It is obvious that conditions (3.10)–(3.13) will not impose any restriction on λ^i_α. First we consider the implications of the lowest component of superfield constraints (3.10)–(3.13). Condition (3.10) implies that

$$G^{i\,j}_{\alpha\beta} = i\epsilon^{ij} f_{\alpha\beta} + 2\epsilon^{ij} \epsilon_{\alpha\beta} D + i\epsilon_{\alpha\beta} \rho^{ij}, \tag{3.15}$$

where $f_{\alpha\beta} = f_{(\alpha\beta)}$ and $\rho^{ij} = \rho^{(ij)}$. Conditions (3.12) and (3.13) further implies the reality condition $D = D^\dagger$ and $\rho^{ij} = \bar{\rho}^{ij}$ where $\bar{\rho}_{ij} = (\rho^{ij})^\dagger$. Condition (3.11) yields

$$H^{i\,j}_{\alpha\dot{\alpha}} = i\epsilon^{ij} h_{\alpha\dot{\alpha}}. \tag{3.16}$$

Higher components of the superfield constraint (3.10)–(3.13) imply further conditions on the fields $f_{\alpha\beta}$, $h_{\alpha\dot{\alpha}}$, and D. One way to realize these conditions is to note that, from equations (3.6), (3.8), and (3.9),

$$\mathscr{F}_{ab} = F_{ab}| = -\tfrac{1}{8} \bar{\sigma}_a{}^{\dot{\alpha}\alpha} \bar{\sigma}_b{}^{\dot{\beta}\beta} \left(\epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta} + \epsilon_{\alpha\beta} \bar{f}_{\dot{\alpha}\dot{\beta}} \right), \tag{3.17}$$

$$F_{a\bar{z}}| = \tfrac{1}{2} h_a, \tag{3.18}$$

$$F_{z\bar{z}}| = 2iD, \tag{3.19}$$

where $\bar{f}_{\dot{\alpha}\dot{\beta}} = (f_{\alpha\beta})^\dagger$ and $h_a = -\frac{1}{2} \sigma_a{}^{\alpha\dot{\alpha}} h_{\alpha\dot{\alpha}}$. The equations (3.9)–(3.13) are the general solution of the Bianchi identities subject to our constraints. Three of the Bianchi identities they

satisfy are

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0, \tag{3.20}$$

$$\partial_a F_{b\bar{z}} + \partial_b F_{\bar{z}a} + \partial_{\bar{z}} F_{ab} = 0, \tag{3.21}$$

$$\partial_a F_{z\bar{z}} + \partial_z F_{\bar{z}a} + \partial_{\bar{z}} F_{az} = 0. \tag{3.22}$$

Taking the lowest component of equation (3.20) gives rise to

$$\partial_a \mathscr{F}_{bc} + \partial_b \mathscr{F}_{ca} + \partial_c \mathscr{F}_{ab} = 0, \tag{3.23}$$

so that $\mathscr{F}_{ab}$ is the field strength of a gauge field V_a. That is

$$\mathscr{F}_{ab} = \partial_a V_b - \partial_b V_a. \tag{3.24}$$

The lowest components of (3.21) and (3.22) give

$$\partial_{[a} h^R_{b]} = -\tfrac{1}{2}(\partial_z + \partial_{\bar{z}})\mathscr{F}_{ab}, \tag{3.25}$$

$$\partial_{[a} h^I_{b]} = -\tfrac{1}{2} i(\partial_z - \partial_{\bar{z}})\mathscr{F}_{ab}, \tag{3.26}$$

$$\partial_a D = -\tfrac{1}{4} i \left\{ (\partial_z - \partial_{\bar{z}}) h^R_a + i(\partial_z + \partial_{\bar{z}}) h^I_a \right\}, \tag{3.27}$$

where $h_a = h^R_a + i h^I_a$.

This is as far as we can go using solely the constraints (3.5). So far we have the following component fields: an SU(2) doublet of spinors λ^i_α, a real gauge field V_a, a complex vector field h_a, a real scalar D and a real SU(2) triplet of scalars ρ^{ij}. We would like to impose further constraints to reduce the number of fields to an irreducible multiplet. As we will show very shortly, it is possible to impose further constraints to ensure that the real and imaginary parts of h_a are each field strengths, one of an anti-symmetric tensor and the other of a scalar. What about the fields D and ρ^{ij}? We will find that D and ρ^{ij} play the role of auxiliary fields. Note that the usual $N = 2$ gauge multiplet has a triplet of auxiliary fields. Hence, if we want to derive the usual gauge multiplet we are led to set D to zero by promoting the constraint (3.12) to the stronger one

$$\epsilon^{\alpha\beta} D^{[i}_\alpha W^{j]}_\beta = 0, \qquad \epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}^{[i}_{\dot{\alpha}} \bar{W}^{j]}_{\dot{\beta}} = 0. \tag{3.28}$$

On the other hand, if we want to derive the vector-tensor multiplet, which has a single auxiliary field, we must eliminate ρ^{ij}. This can be achieved by promoting condition (3.13) to the stronger condition

$$\epsilon^{\alpha\beta} D^{(i}_\alpha W^{j)}_\beta = 0, \qquad \epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}^{(i}_{\dot{\alpha}} \bar{W}^{j)}_{\dot{\beta}} = 0. \tag{3.29}$$

Henceforth, we add condition (3.29) as a further constraint on the theory. Now let us proceed using conditions (3.10)–(3.12) as well as the stronger version of (3.13), constraint

(3.29). Apart from eliminating ρ^{ij}, it can be shown that the consequences of imposing (3.29) are

$$\begin{aligned} \partial^a h^R_a &= i(\partial_z - \partial_{\bar z})D, \\ \partial^a h^I_a &= -(\partial_z + \partial_{\bar z})D, \\ \partial^b \mathscr{F}_{ab} &= -\tfrac{1}{4}\left\{(\partial_z + \partial_{\bar z})h^R_a + i(\partial_z - \partial_{\bar z})h^I_a\right\}. \end{aligned} \tag{3.30}$$

We note that the conditions obtained so far on h_a given in (3.26) and (3.30) are not sufficient to render the real and imaginary components of h_a as field strengths. We would like, therefore, to impose yet another, and final, constraint that will achieve this goal. We take a reality condition on the central charge

$$\partial_z W^i_\alpha = \partial_{\bar z} W^i_\alpha. \tag{3.31}$$

This means that superfield W^i_α, and hence all the curvature tensor components, depend on the two central charges z and $\bar z$ only through the combination $z + \bar z$ with no dependence on $z - \bar z$. Substituting from constraint (3.31) into (3.26) and (3.30) immediately gives the following constraints on the real and imaginary components of h_a

$$\partial^a h^R_a = 0, \qquad \partial_{[a} h^I_{b]} = 0. \tag{3.32}$$

These equations achieve just the goal we were aiming for. They assert that h^R_a and h^I_a are field strengths of an anti-symmetric tensor and a scalar field respectively. That is

$$\begin{aligned} h^R_a &= \tfrac{1}{3}\epsilon_{abcd} H^{bcd} = \tfrac{1}{3}\epsilon_{abcd}\partial^b B^{cd}, \\ h^I_a &= 2\partial_a \phi. \end{aligned} \tag{3.33}$$

In conclusion, we find that the component fields in W^i_α are exactly those of the vector-tensor multiplet, namely $(\lambda^i_\alpha, \phi, B_{ab}, V_a, D)$. It will not have escaped the readers notice that these fields are actually functions of the central charge coordinates as well as the spacetime coordinates. However, by virtue of constraint (3.31), they depend on the central charge coordinates only via $z + \bar z$. Let us introduce a real central charge coordinate $s = z + \bar z$. Moreover, the dependence of these fields on s is highly restricted by the conditions we derived so far. In fact, as we now show, these conditions completely determine their dependence on s in terms of their lowest component in their central charge expansion. To see this let us summarize the relevant conditions in (3.26), (3.27), and (3.30) derived on the bosonic fields. They are

$$\begin{aligned} \partial_s D &= -\tfrac{1}{2}\partial^a h^I_a, \\ \partial_s h^R_a &= -2\partial^b \mathscr{F}_{ab}, \\ \partial_s h^I_a &= 2\partial_a D, \\ \partial_s \mathscr{F}_{ab} &= -\partial_{[a} h^R_{b]}. \end{aligned} \tag{3.34}$$

Equations (3.34) constitute a system of first-order "differential equations" for the s dependence of the bosonic fields. The general solution is completely determined in terms of the "initial conditions"; that is, the values of the fields at $s = 0$, say, $D(x^m)$, $h_a(x^m)$, and $\mathscr{F}_{ab}(x^m)$. Thus in a Taylor series expansion in s each term in the series gets related to lower-order terms, so that for instance,

$$D(x^m, s) = D(x^m) - \tfrac{1}{2}\partial^a h^I_a(x^m)s - \tfrac{1}{2}\Box D(x^m)s^2 + \cdots , \tag{3.35}$$

with similar formulas for h^R_a, h^I_a, and $\mathscr{F}_{ab}$. Furthermore, in the next section, we will derive the corresponding equation for the fermionic field λ^i_α.

4. Supersymmetry and Central Charge Transformations

In this section, we would like to derive the supersymmetry and central charge transformations of the component fields. In order to exhibit these transformations it is necessary to compute the part of W^i_α quadratic in the fermionic coordinates. Moreover, since all the component fields are present in the lowest and first order terms of W^i_α, the quadratic term is sufficient for computing the supersymmetry transformations. We have

$$W^i_\alpha = \lambda^i_\alpha + i\theta^{\beta i} f_{\alpha\beta} + 2\theta^i_\alpha D + i\bar\theta^{\dot\alpha i} h_{\alpha\dot\alpha} + \theta^\beta_j\theta^\gamma_k \Lambda^{i\,j\,k}_{\alpha\beta\gamma} + \theta^\beta_j\bar\theta^{\dot\gamma}_k \Pi^{i\,j\,k}_{\alpha\beta\dot\gamma} + \bar\theta^{\dot\beta}_j\bar\theta^{\dot\gamma}_k \Sigma^{i\,j\,k}_{\alpha\dot\beta\dot\gamma} + \mathcal{O}(\theta^3) \tag{4.1}$$

Let us first consider the constraint equations (3.10), (3.11), and (3.29). Demanding that the linear term in θ in these equations vanish completely determines Λ, Π, and Σ. We find

$$\begin{aligned}
\Lambda^{i\,j\,k}_{\alpha\beta\gamma} &= \tfrac{1}{2}i\epsilon_{\beta\gamma}\epsilon^{ij}\partial_s\lambda^k_\alpha + \tfrac{1}{2}i\epsilon_{\beta\gamma}\epsilon^{ik}\partial_s\lambda^j_\alpha,\\
\Pi^{i\,j\,k}_{\alpha\beta\dot\gamma} &= \tfrac{1}{2}i\epsilon^{jk}\sigma^a{}_{\beta\dot\gamma}\partial_a\lambda^i_\alpha - \tfrac{1}{2}i\epsilon^{ij}\sigma^a{}_{\beta\dot\gamma}\partial_a\lambda^k_\alpha - \tfrac{3}{2}i\epsilon^{ik}\sigma^a{}_{\beta\dot\gamma}\partial_a\lambda^j_\alpha,\\
\Sigma^{i\,j\,k}_{\alpha\dot\beta\dot\gamma} &= \tfrac{1}{2}i\epsilon_{\dot\beta\dot\gamma}\epsilon^{ij}\partial_s\lambda^k_\alpha + \tfrac{1}{2}i\epsilon_{\dot\beta\dot\gamma}\epsilon^{ik}\partial_s\lambda^j_\alpha.
\end{aligned} \tag{4.2}$$

We note that these terms do not involve any new fields. They are completely given in terms of the lowest component of W^i_α, namely λ^i_α. We are still left with equation (3.12) which is a reality condition. Evaluating the linear term in (3.12), and making use of (4.2), yields the following constraint on the fermionic field λ^i_α

$$\partial_s\lambda^i_\alpha = \sigma^a{}_{\alpha\dot\alpha}\partial_a\bar\lambda^{\dot\alpha i}, \tag{4.3}$$

where $\bar\lambda_{\dot\alpha i} = (\lambda^i_\alpha)^\dagger$. Equation (4.3) is the relation for the fermionic field λ^α_i corresponding to equations (3.34) for the bosonic fields. It fixes the expansion of λ^i_α in the central charge s, leaving only the lowest component $\lambda^i_\alpha(x^m)$ arbitrary.

We are now in a position to compute the supersymmetry transformation of the different component fields. To do this we have to act on W^i_α with

$$\delta_\xi = \xi^\alpha_i Q^i_\alpha + \bar\xi^i_{\dot\alpha}\bar Q^{\dot\alpha}_i. \tag{4.4}$$

We find

$$\begin{aligned}
\delta_\xi D &= -\tfrac{1}{2} i \left(\xi_i^\alpha \sigma^a{}_{\alpha\dot\alpha} \partial_a \bar\lambda^{\dot\alpha i} + \bar\xi_i^{\dot\alpha} \sigma^a{}_{\alpha\dot\alpha} \partial_a \lambda^{\alpha i} \right), \\
\delta_\xi \phi &= \tfrac{1}{2} i \left(\xi_{\alpha i} \lambda^{\alpha i} - \bar\xi_{\dot\alpha i} \bar\lambda^{\dot\alpha i} \right), \\
\delta_\xi B_{cd} &= \tfrac{1}{6} \epsilon_{abcd} \sigma^{ab}{}_{(\alpha\beta)} \xi_i^\alpha \lambda^{\beta j} + \tfrac{1}{6} \epsilon_{abcd} \bar\sigma^{ba}{}_{(\dot\alpha\dot\beta)} \bar\xi_i^{\dot\alpha} \bar\lambda^{\dot\beta i}, \\
\delta_\xi V_a &= -\tfrac{1}{2} \xi_{\alpha i} \bar\sigma_a{}^{\dot\alpha\alpha} \bar\lambda^k_{\dot\alpha} + \tfrac{1}{2} \bar\xi_{\dot\alpha i} \bar\sigma_a{}^{\dot\alpha\alpha} \lambda^k_\alpha, \\
\delta_\xi \lambda^i_\alpha &= 2i \xi^{\beta i} \epsilon^{\dot\alpha\dot\beta} \sigma^a{}_{\alpha\dot\alpha} \sigma^b{}_{\beta\dot\beta} \partial_{[a} V_{b]} + 2 \xi^i_\alpha D + \tfrac{i}{3} \bar\xi^{\dot\alpha i} \sigma^a{}_{\alpha\dot\alpha} \epsilon_{abcd} \partial^b B^{cd} - 2 \sigma^a{}_{\alpha\dot\alpha} \bar\xi^{\dot\alpha} \partial_a \phi.
\end{aligned} \tag{4.5}$$

Note that the fields in these supersymmetry transformations are formally functions of both x^m and s. We have already shown that the only independent components of these fields are the lowest order terms in s. Further, transformations (4.5) do not mix component fields from different orders in s. Thus equations (4.5) also express the variations of the lowest-order independent fields, which are functions of x^m only.

To get the central charge transformations we have to act with $\delta_\omega = \omega \partial_s$ on W^i_α. It is straightforward to get the following set of transformations

$$\begin{aligned}
\delta_\omega D &= -\omega \Box \phi, \\
\delta_\omega \phi &= \omega D, \\
\delta_\omega B^{cd} &= 3 \omega \epsilon^{abcd} \partial_a V_b, \\
\delta_\omega V_a &= -\tfrac{1}{6} \omega \epsilon_{abcd} \partial^b B^{cd}, \\
\delta_\omega \lambda^i_\alpha &= \omega \sigma^a{}_{\alpha\dot\alpha} \partial_a \bar\lambda^{\dot\alpha i}.
\end{aligned} \tag{4.6}$$

Expressions (4.5) and (4.6) reproduce the supersymmetry and central charge transformations of the vector-tensor multiplet given in the component field calculations of [5-7]. They are a realization of the $N = 2$ supersymmetry algebra with central charge. Note that as a result of our constraints, in particular (3.31), only one of the two central charges is represented nontrivially.

We now discuss how to write a superfield action for the vector-tensor multiplet. We start by noting that the relevant field strengths of the different bosonic fields occur at the linear level in W^i_α. To get a quadratic expression in these field strengths, we have to consider expressions quadratic in W^i_α. These expressions can contain both D^i_α and $\bar D_{\dot\alpha i}$. Contracting the indices to get a proper Lorentz invariant Lagrangian implies that there must be an even number of D's. In such an expression, the relevant terms will occur at θ^2 level if it involves no derivatives or the lowest component if it involves two derivatives. In the former case we will have to integrate over two θ's only. But in contrast to the chiral space for $N = 1$ supersymmetry, there is no two-dimensional subspace of the $N = 2$ superspace over which supersymmetry is represented. Hence, we are forced to consider expressions quadratic in W^i_α with two supersymmetric covariant derivatives, and to integrate over spacetime only. In fact, we find that the free Lagrangian for all the component fields is contained in the lowest

component of

$$\mathscr{L} = -\tfrac{1}{192} D_i^\alpha D_{\alpha j} W^{\beta i} W_\beta^j + \tfrac{1}{192} \bar{D}_{\dot\alpha i} \bar{D}_j^{\dot\alpha} W^{\beta i} W_\beta^j + \text{h.c.} \tag{4.7}$$

To be precise, we have

$$\mathscr{L}|_{\theta=s=0} = -\tfrac{1}{4}\mathscr{F}_{ab}\mathscr{F}^{ab} - \tfrac{1}{2}\partial_a\phi\partial^a\phi - \tfrac{1}{12}\partial_a B_{bc}\partial^a B^{bc} + \tfrac{1}{2}D^2 - \tfrac{1}{4}i\lambda^{\alpha i}\sigma^a{}_{\alpha\dot\alpha}\partial_a\bar\lambda_i^{\dot\alpha}. \tag{4.8}$$

It is straightforward to verify that Lagrangian (4.8) is invariant under supersymmetry transformations (4.5) up to a total divergence. Although this implies that it must be invariant under central charge transformation as well, since the latter is equivalent to two successive supersymmetry transformations, one can check directly that it is indeed invariant under (4.6) up to a total divergence. This invariance of the lowest component of the Lagrangian under central charge and supersymmetry transformations, in fact, implies that the higher components are spacetime total divergences. To show this let us consider first the invariance under central charge transformation. Since the Lagrangian transforms by a total divergence under a single central charge transformation, it must does so under n such transformations. That is

$$\begin{aligned}
\delta_\omega^n \mathscr{L}|_{\theta=s=0} &= \left\{\omega^n \frac{\partial^n}{\partial s^n}\mathscr{L}\right\}\Bigg|_{\theta=s=0} \\
&= \left\{\omega^n \frac{\partial^n}{\partial s^n}\mathscr{L}|_{\theta=0}\right\}\Bigg|_{s=0} \\
&= \omega^n \partial^a M_{n,a}(x^m).
\end{aligned} \tag{4.9}$$

Put another way, all the coefficients in the s expansion of $\mathscr{L}|_{\theta=0}$ but the lowest one are, in fact, spacetime total divergences. Therefore we have

$$\mathscr{L}|_{\theta=0} = \mathscr{L}|_{\theta=s=0} + \sum_{n=1}^{\infty} \partial^a M_{n,a}\frac{s^n}{n!}. \tag{4.10}$$

Although a little involved we can extend this proof to the expansion in θ as well. That is, it can be shown that

$$\mathscr{L} = \mathscr{L}|_{\theta=s=0} + \partial^a \mathscr{M}_a(z^M). \tag{4.11}$$

We would like to stress that this is completely general. If the lowest component of a superfield is supersymmetric invariant up to a spacetime total divergence, all the higher components in s as well as θ are spacetime total divergences. This has the immediate consequence that a spacetime integral of this superfield will single out the lowest component. In other words we can write down the following superfield expression for the invariant action

$$\begin{aligned}
S &= \int d^4x \left\{-\tfrac{1}{192} D_i^\alpha D_{\alpha j} W^{\beta i} W_\beta^j + \tfrac{1}{192} \bar{D}_{\dot\alpha i} \bar{D}_j^{\dot\alpha} W^{\beta i} W_\beta^j + \text{h.c.}\right\} \\
&= \int d^4x \left\{-\tfrac{1}{4}\mathscr{F}_{ab}\mathscr{F}^{ab} - \tfrac{1}{2}\partial_a\phi\partial^a\phi - \tfrac{1}{12}\partial_a B_{bc}\partial^a B^{bc} + \tfrac{1}{2}D^2 - \tfrac{1}{4}i\lambda^{\alpha i}\sigma^a{}_{\alpha\dot\alpha}\partial_a\bar\lambda_i^{\dot\alpha}\right\}.
\end{aligned} \tag{4.12}$$

This represents a central charge generalization of the type of superactions considered in [10]. This completes our study of the superfield formulation of the vector-tensor multiplet.

5. Concluding Remarks

We conclude by briefly mentioning the connection between the superspace formulation of the vector-tensor multiplet and $N = 1$ supersymmetry in six dimensions. The presence of central charge led us to augment $N = 2$ superspace by two new bosonic coordinates, z and $\bar{z}$. The resulting space, with six bosonic coordinates, is actually none-other than the superspace of $N = 1$ supersymmetry in six dimensions (though with the fermionic coordinates treated slightly unconventionally, being formulated as a pair of SU(2) Majorana-Weyl spinors [14-16]). The introduction of a connection A in the central charge superspace then mirrors the formulation of supersymmetric gauge theory in six dimensions. In fact our first set of constraints (3.5) are precisely those used for the six-dimensional vector multiplet [15, 16]. To obtain the vector-tensor multiplet we imposed two further constraints. The first condition (3.29) still has a six-dimensional interpretation: it corresponds to the on-shell condition for the six- dimensional vector multiplet. It is important to note that this does not imply that the multiplet is on-shell in four-dimensions. Instead it implies a relationship between the derivatives of the component fields in the internal z–$\bar{z}$ directions and in the four-dimensional x^a directions, as shown in equation (3.30). The second condition (3.31) breaks the six-dimensional Lorentz symmetry. It forces the superfield to be independent of one of the internal z–$\bar{z}$ directions. As we have shown, together these conditions are sufficient to completely determine the dependence of the multiplet on the central charge coordinates. The connection to the component formulation of the vector-tensor multiplet given by Sohnius *et al.* [6] now also becomes clear. The second condition (3.31) is a conventional dimensional reduction of the six- dimensional vector multiplet down to five-dimensions. The first condition (3.29), corresponding to an on-shell condition in six- dimensions, reproduces the "dimensional reduction by Legendre transform" from five to four dimensions.

A superspace formulation of the vector-tensor multiplet allows a number of further issues to be addressed. First is the question of duality. On shell, the usual vector multiplet and the vector-tensor multiplet describe the same degrees of freedom. From the $N = 1$ point of view, one is the dual of the other since the chiral multiplet in the usual vector multiplet is dual to the tensor multiplet in the vector-tensor multiplet. Central charge superspace should provide a realization of this duality in a manifestly $N = 2$ supersymmetric way. A related question is to find an unconstrained $N = 2$ prepotential for the vector-tensor multiplet. We also note that the form of the multiplet described here has been explicitly non-interacting. However, the superspace formalism also allows the description of non-Abelian multiplets as well as multiplets with gauged central charge and Chern-Simons terms as have been discussed in components in [8]. It is also pertinent to the discussion of the most general form of the

action for the vector-tensor multiplet. Finally the central charge superspace can be used to give a superfield description of other exotic $N = 2$ multiplets, in particular the tensor and double-tensor multiplets mentioned in [8]. We leave the discussion of these issues to future publications.

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